On Greenberg's L-invariant of the symmetric sixth power of an ordinary cusp form

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Abstract

We derive a formula for Greenberg's L-invariant of Tate twists of the symmetric sixth power of an ordinary non-CM cuspidal newform of weight ≥ 4, under some technical assumptions. This requires a "sufficiently rich" Galois deformation of the symmetric cube which we obtain from the symmetric cube lift to $\mathrm{GSp}(4)_{/\mathbf{Q}}$ of Ramakrishnan–Shahidi and the Hida theory of this group developed by Tilouine–Urban. The L-invariant is expressed in terms of derivatives of Frobenius eigenvalues varying in the Hida family. Our result suggests that one could compute Greenberg's L-invariant of all symmetric powers by using appropriate functorial transfers and Hida theory on higher rank groups.

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Introduction

The notion of an L-invariant was introduced by Mazur, Tate, and Teitelbaum in their investigations of a p-adic analogue of the Birch and Swinnerton-Dyer conjecture in [MTT86]. When considering the p-adic L-function of an elliptic curve E over $\mathbf{Q}$ with split, multiplicative reduction at p, they saw that its p-adic L-function vanishes even when its usual L-function does not (an "exceptional zero" or "trivial zero"). They introduced a p-adic invariant, the "(p-adic) L-invariant", of E as a fudge factor to recuperate the p-adic interpolation property of $L(1, E, \chi)$ using the *derivative* of its p-adic L-function. Their conjecture appears in [MTT86, §§13–14] and was proved by Greenberg and Stevens in [GS93]. The proof conceptually splits up into two parts. One part relates the L-invariant of E to the derivative in the "weight direction" of the unit eigenvalue of Frobenius in the Hida family containing f (the modular form corresponding to E). The other part uses the functional equation of the two-variable p-adic L-function to relate the derivative in the weight direction to the derivative of interest, in the "cyclotomic direction". In this paper, we provide an analogue of the first part of this proof replacing the p-adic Galois representation ρ_f attached to f with Tate twists of $\mathrm{Sym}^6\rho_f$. More specifically, we obtain a formula for Greenberg's L-invariant ([G94]) of Tate twists of $\mathrm{Sym}^6\rho_f$ in terms of derivatives in weight directions of the unit eigenvalues of Frobenius varying in some ordinary Galois deformation of $\mathrm{Sym}^3\rho_f$.

Let us describe the previous work in this subject. In his original article, Greenberg ([G94]) computes his L-invariant for all symmetric powers of ρ_f when f is associated to an elliptic curve with split, multiplicative reduction at p. In this case, the computation is local, and quite simple. In a series of articles, Hida has relaxed the assumption on f allowing higher weights and dealing with Hilbert modular forms (see [Hi07]), but still requiring, for the most part, ρ_f to be (potentially) non-cristalline (though semistable) at p in order to obtain an explicit formula for the L-invariant. A notable exception where a formula is known in the cristalline case is the symmetric square, done by Hida in [Hi04] (see also chapter 2 of the author's Ph.D. thesis [H-PhD] for a slightly different approach). Another exception comes again from Greenberg's original article ([G94]) where he computes his L-invariant when E has good ordinary reduction at p *and has complex multiplication.* In this case, the symmetric powers are reducible and the value of the L-invariant comes down to the result of Ferrero–Greenberg ([FeG78]). The general difficulty in the cristalline case is that Greenberg's L-invariant is then a *global* invariant and its computation requires the construction of a global Galois cohomology class.

In this article, we attack the cristalline case for the next symmetric power which has an L-invariant, namely the sixth power (a symmetric power n has an L-invariant in the cristalline case only when $n \equiv 2 \pmod 4$). In general, one could expect to be able to compute Greenberg's L-invariant of $\mathrm{Sym}^n\rho_f$ by looking at ordinary Galois deformations of $\mathrm{Sym}^{\frac{n}{2}}\rho_f$ (see §1.3). Unfortunately, when $n > 2$ in the cristalline case, the $\mathrm{Sym}^{\frac{n}{2}}$ of the Hida deformation of ρ_f is insufficient. The new ingredient we bring to the table is the idea to use a functorial transfer of $\mathrm{Sym}^{\frac{n}{2}}f$ to a higher rank group, use Hida theory there, and hope that the additional variables in the Hida family provide non-trivial Galois cohomology classes. In theorem A, we show that this works for $n = 6$ using the symmetric cube lift of Ramakrishnan–Shahidi ([RS07]) (under certain technical assumptions). This provides hope that such a strategy would yield formulas for Greenberg's L-invariant for all symmetric powers in the cristalline case. The author is currently investigating if the combined use of the potential automorphy results of [BLGHT09], the functorial descent to a unitary group,

and Hida theory on it ([Hi02]) will be of service in this endeavour.

We also address whether the L-invariant of the symmetric sixth power equals that of the symmetric square. There is a guess, due to Greenberg, that it does. We fall short of providing a definitive answer, but obtain a relation between the two in theorem B.

There are several facets of the symmetric sixth power L-invariant which we do not address. We do not discuss the expected non-vanishing of the L-invariant nor its expected relation to the size of a Selmer group. Furthermore, we make no attempt to show that Greenberg's L-invariant is the *actual* L-invariant appearing in an interpolation formula of L-values. Aside from the fact that the p-adic L-function of the symmetric sixth power has not been constructed, a major impediment to proving this identity is that the point at which the p-adic L-function has an exceptional zero is no longer the centre of the functional equation, and a direct generalization of the second part of the proof of Greenberg–Stevens is therefore not possible. Citro suggests a way for dealing with this latter problem in the symmetric square case in [Ci08]. Finally, we always restrict to the case where f is ordinary at p. Recently, in [Be09], Benois has generalized Greenberg's definition of L-invariant to the non-ordinary case, and our results suggest that one could hope to compute his L-invariant using the eigenvariety for $\mathrm{GSp}(4)_{/\mathbf{Q}}$.

We remark that the results of this article were obtained in the author's Ph.D. thesis ([H-PhD, Chapter 3]). There, we give a slightly different construction of the global Galois cohomology class, still using the same deformation of the symmetric cube. In particular, we use Ribet's method of constructing a global extension of Galois representations by studying an irreducible, but residually reducible, representation. We refer to [H-PhD] for details.

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Notation and conventions

We fix throughout a prime $p \geq 3$ and an isomorphism $\iota_\infty : \overline{\mathbf{Q}}_p \cong \mathbf{C}$. For a field F, G_F denotes the absolute Galois group of F. We fix embeddings ι_ℓ of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_\ell$ for all primes ℓ. These define primes $\overline{\ell}$ of $\overline{\mathbf{Q}}$ over ℓ, and we let G_ℓ denote the decomposition group of $\overline{\ell}$ in $G_{\mathbf{Q}}$, which we may thus identify with $G_{\mathbf{Q}_\ell}$. Let I_ℓ denote the inertia subgroup of G_ℓ. Let $\mathbf{A}$ denote the adeles of $\mathbf{Q}$ and let $\mathbf{A}_f$ be the finite adeles.

By a p-adic representation (over K) of a topological group G, we mean a continuous representation $\rho : G \to \mathrm{Aut}_K(V)$, where K is a finite extension of $\mathbf{Q}_p$ and V is a finite-dimensional K-vector space equipped with its p-adic topology. Let χ_p denote the p-adic cyclotomic character. We denote the Tate dual $\mathrm{Hom}(V, K(1))$ of V by V^*. Denote the Galois cohomology of the absolute Galois group of F with coefficients in M by $H^i(F, M)$.

For compatibility with [G94], we take Frob_p to be an *arithmetic* Frobenius element at p, and we normalize the local reciprocity map $\mathrm{rec} : \mathbf{Q}_p^\times \to G_{\mathbf{Q}_p}^{\mathrm{ab}}$ so that Frob_p corresponds to p. We normalize the p-adic logarithm $\log_p : \overline{\mathbf{Q}}_p^\times \longrightarrow \overline{\mathbf{Q}}_p$ by $\log_p(p) = 0$.

1 Greenberg's theory of trivial zeroes

In [G94], Greenberg introduced a theory describing the expected order of the trivial zero, as well as a conjectural value for the L-invariant of a p-ordinary motive. In this section, we briefly describe this theory, restricting ourselves to the case we will require in the sequel, specifically, we will assume the "exceptional subquotient" W is isomorphic to the trivial representation. We end this section by explaining our basic method of computing L-invariants of symmetric powers of cusp forms.

1.1 Ordinarity, exceptionality, and some Selmer groups

Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}(V)$ be a p-adic representation over a field K. Recall that V is called *ordinary* if there is a descending filtration $\{F^iV\}_{i\in\mathbf{Z}}$ of G_p-stable K-subspaces of V such that I_p acts on $\mathrm{gr}^iV = F^iV/F^{i+1}V$ via multiplication by χ_p^i (and $F^iV = V$ (resp. $F^iV = 0$) for i sufficiently negative (resp. sufficiently positive)). Under this assumption, Greenberg ([G89]) has defined what we call the *ordinary Selmer group* for V as

$$\mathrm{Sel}_{\mathbf{Q}}(V) := \ker\left(H^1(\mathbf{Q},V) \longrightarrow \prod_v H^1(\mathbf{Q}_v,V)\,/L_v(V)\right)$$

where the product is over all places v of $\mathbf{Q}$ and the local conditions $L_v(V)$ are given by

$$L_v(V) := \begin{cases} H^1_{\mathrm{nr}}(\mathbf{Q}_v,V) := \ker\big(H^1(\mathbf{Q},V) \to H^1(I_v,V)\big), & v \neq p \\ H^1_{\mathrm{ord}}(\mathbf{Q}_p,V) := \ker\big(H^1(\mathbf{Q},V) \to H^1\big(I_p, V/F^1V\big)\big), & v = p. \end{cases} \tag{1.1}$$

This Selmer group is conjecturally related to the p-adic L-function of V at $s = 1$.

To develop the theory of exceptional zeroes following Greenberg ([G94]), we introduce three additional assumptions on V (which will be satisfied by the V in which we are interested). Assume

(C) V is *critical* in the sense that $\dim_K V/F^1V = \dim_K V^-$, where V^- is the (-1)-eigenspace of complex conjugation,

(U) V has no G_p subquotient isomorphic to the cristalline extension of K by $K(1)$,

(S) G_p acts semisimply on gr^iV for all $i \in \mathbf{Z}$.

If V arises from a motive, condition (C) is equivalent to that motive being critical at $s = 1$ in the sense of Deligne [D79] (see [G89, §6]). Condition (U) will come up when we want to define the L-invariant. Assumption (S) allows us to refine the ordinary filtration and define a G_p-subquotient of V that (conjecturally) regulates the behaviour of V with respect to exceptional zeroes.

Definition 1.1.

(a) Let $F^{00}V$ be the maximal G_p-subspace of F^0V such that G_p acts trivially on $F^{00}V/F^1V$.

(b) Let $F^{11}V$ be the minimal G_p-subspace of F^1V such that G_p acts on $F^1V/F^{11}V$ via multiplication by χ_p.

(c) Define the *exceptional subquotient* W of V as

$$W := F^{00}V/F^{11}V.$$

(d) V is called *exceptional* if $W \neq 0$.

Note that W is ordinary with $F^2W = 0, F^1W = F^1/F^{11}V$, and $F^0W = W$. For $? = 00, 11$, or $i \in \mathbf{Z}$, we denote

$$F^? H^1(\mathbf{Q}_p, V) := \mathrm{im}\Big(H^1\Big(\mathbf{Q}_p, F^? V\Big) \longrightarrow H^1(\mathbf{Q}_p, V)\Big).$$

For simplicity, we impose the following condition on V which will be sufficient for our later work:

(T′) $W \cong K$, i.e. $F^{11}V = F^1V$ and $\dim_K F^{00}V/F^1V = 1$.

We remark that this is a special case of condition (T) of [G94].

The ordinarity of V and assumptions (C), (U), (S), and (T′) allow us to introduce Greenberg's *balanced Selmer group* $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$ of V (terminology due to Hida) as follows. The local conditions $\overline{L}_v(V)$ at $v \neq p$ are simply given by the unramified conditions $L_v(V)$ of (1.1). At p, $\overline{L}_p(V)$ is characterized by the following two properties:

(Bal1) $F^{11}H^1(\mathbf{Q}_p, V) \subseteq \overline{L}_p(V) \subseteq F^{00}H^1(\mathbf{Q}_p, V)$,

(Bal2) $\mathrm{im}\big(\overline{L}_p(V) \to H^1(\mathbf{Q}_p, W)\big) = H^1_{\mathrm{nr}}(\mathbf{Q}_p, W)$.

The balanced Selmer group of V is

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) := \ker\left(H^1(\mathbf{Q}, V) \longrightarrow \prod_v H^1(\mathbf{Q}_v, V)\,/\overline{L}_v(V)\right).$$

The rationale behind the name "balanced" is provided by the following basic result of Greenberg's.

Proposition 1.2 (Proposition 2 of [G94]). *The balanced Selmer groups of V and V^* have the same dimension.*

To make the reader feel more familiar with the balanced Selmer group, we offer the following result on its value under our running assumptions.

Proposition 1.3. *Let V be an ordinary p-adic representation of $G_{\mathbf{Q}}$. Under assumptions* (C), (U), (S), *and especially* (T′), *we have the following equalities*

$$\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) = \mathrm{Sel}_{\mathbf{Q}}(V) = H^1_g(\mathbf{Q}, V) = H^1_f(\mathbf{Q}, V)$$

where $H^1_g(\mathbf{Q}, V)$ and $H^1_f(\mathbf{Q}, V)$ are the Bloch–Kato Selmer groups introduced in [BK90].

Proof. The second equality is due to Flach ([Fl90, Lemma 2]) and the last equality follows from [BK90, Corollary 3.8.4]. We proceed to prove the first equality. The local conditions at $v \neq p$ are the same for $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)$ and $\mathrm{Sel}_{\mathbf{Q}}(V)$ so we are left to show that $\overline{L}_p(V) = L_p(V)$.

Let $c \in \overline{L}_p(V)$. Condition (Bal1) implies that there is $c' \in H^1\big(\mathbf{Q}_p, F^{00}V\big)$ mapping to c. By (Bal2), the image of c' under the map in the bottom row of the commutative diagram

$$\begin{tikzcd}
H^1(\mathbf{Q}_p, V) \arrow[r] & H^1\big(I_p, V/F^1V\big) \\
H^1\big(\mathbf{Q}_p, F^{00}V\big) \arrow[r] \arrow[u] & H^1(I_p, W) \arrow[u]
\end{tikzcd}$$

is zero. Thus, c is in the kernel of the map in the top row, which is exactly $L_p(V)$.

For the reverse equality, let $c \in L_p(V)$ and consider the commutative diagram

$$\begin{tikzcd}
H^1\big(\mathbf{Q}_p, V/F^{00}V\big) \arrow[r, "f_2"] & H^1\big(I_p, V/F^{00}V\big) \\
c \in H^1(\mathbf{Q}_p, V) \arrow[r, "f_1"] \arrow[u, "f_3"] & H^1\big(I_p, V/F^1V\big) \arrow[u] \\
H^1\big(\mathbf{Q}_p, F^{00}V\big) \arrow[r] \arrow[u] & H^1(I_p, W)\,. \arrow[u, "f_4"]
\end{tikzcd}$$

The local condition $L_p(V)$ satisfies (Bal1) if $c \in \ker f_3$. By definition, $c \in \ker f_1$, so we show that $\ker f_2 = 0$. By inflation-restriction, $\ker f_2$ is equal to

$$\operatorname{im}\left(H^1\Big(G_p/I_p, \big(V/F^{00}V\big)^{I_p}\Big) \longrightarrow H^1\big(\mathbf{Q}_p, V/F^{00}V\big)\right).$$

Note that $\big(V/F^{00}V\big)^{I_p} = F^0V/F^{00}V$. The pro-cyclic group G_p/I_p has (topological) generator Frob_p, so

$$H^1\big(G_p/I_p, F^0V/F^{00}V\big) \cong (F^0V/F^{00}V)/\left((\mathrm{Frob}_p - 1)(F^0V/F^{00}V)\right) = 0$$

where the last equality is because $F^{00}V$ was defined to be exactly the part of F^0V on which Frob_p acts trivially (mod F^1V). Thus, $L_p(V)$ satisfies (Bal1), so there is a $c' \in H^1\big(\mathbf{Q}_p, F^{00}V\big)$ mapping to c. Its image in $H^1\big(I_p, V/F^1V\big)$ is trivial, so it suffices to show that $\ker f_4 = 0$ to conclude that $L_p(V)$ satisfies (Bal2). By the long exact sequence in cohomology, the exactness (on the right) of

$$0 \longrightarrow W^{I_p} \longrightarrow (V/F^1V)^{I_p} \longrightarrow (V/F^{00}V)^{I_p} \longrightarrow 0$$

shows that $\ker f_4 = 0$. $\square$

Remark 1.4. In fact, this result is still valid if (T′) is relaxed to simply $F^{11}V = F^1V$ (see [H-PhD, Lemma 1.3.4]).

1.2 Greenberg's L-invariant

We now proceed to define Greenberg's L-invariant. To do so, we impose one final condition on V, namely

(Z) the balanced Selmer group of V is zero: $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) = 0$.

This will allow us to define a one-dimensional global subspace $H^{\mathrm{exc}}_{\mathrm{glob}}$ in a global Galois cohomology group (via some local conditions) whose image in $H^1(\mathbf{Q}_p, W)$ will be a line. The slope of this line is the L-invariant of V.

Let Σ denote the set of primes of $\mathbf{Q}$ ramified for V, together with p and ∞, let $\mathbf{Q}_\Sigma$ denote the maximal extension of $\mathbf{Q}$ unramified outside Σ, and let $G_\Sigma := \mathrm{Gal}(\mathbf{Q}_\Sigma/\mathbf{Q})$. By definition, $\overline{\mathrm{Sel}}_{\mathbf{Q}}(V) \subseteq H^1(G_\Sigma, V)$. The Poitou–Tate exact sequence with local conditions $\overline{L}_v(V)$ yields the exact sequence

$$0 \longrightarrow \overline{\mathrm{Sel}}_{\mathbf{Q}}(V) \longrightarrow H^1(G_\Sigma, V) \longrightarrow \bigoplus_{v\in\Sigma} H^1(\mathbf{Q}_v, V)\,/\overline{L}_v(V) \longrightarrow \overline{\mathrm{Sel}}_{\mathbf{Q}}(V^*).$$

Combining this with assumption (Z) and proposition 1.2 gives an isomorphism

$$H^1(G_\Sigma, V) \cong \bigoplus_{v\in\Sigma} H^1(\mathbf{Q}_v, V)\,/\overline{L}_v(V). \tag{1.2}$$

Definition 1.5. Let $H^{\mathrm{exc}}_{\mathrm{glob}}$ be the one-dimensional subspace[1] of $H^1(G_\Sigma, V)$ corresponding to the subspace $F^{00}H^1(\mathbf{Q}_p, V)\,/\overline{L}_p(V)$ of $\bigoplus_{v\in\Sigma} H^1(\mathbf{Q}_v, V)\,/\overline{L}_v(V)$ under the isomorphism in (1.2).

By definition of $F^{00}V$, we know that $(V/F^{00}V)^{G_p} = 0$. Hence, we have injections

$$H^1\big(\mathbf{Q}_p, F^{00}V\big) \hookrightarrow H^1(\mathbf{Q}_p, V)$$

and

$$H^1(\mathbf{Q}_p, W) \hookrightarrow H^1\big(\mathbf{Q}_p, V/F^1V\big)\,.$$

Definition 1.6. Let $H^{\mathrm{exc}}_{\mathrm{loc}} \subseteq H^1(\mathbf{Q}_p, W)$ be the image of $H^{\mathrm{exc}}_{\mathrm{glob}}$ in the bottom right cohomology group in the commutative diagram

$$\begin{tikzcd}
H^1(G_\Sigma, V) \arrow[r] & H^1(\mathbf{Q}_p, V) \arrow[r] & H^1\big(\mathbf{Q}_p, V/F^1V\big) \\
\rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \\
H^{\mathrm{exc}}_{\mathrm{glob}} \arrow[r] & F^{00}H^1(\mathbf{Q}_p, V) & \\
& \rotatebox{90}{$\cong$} & \\
& H^1\big(\mathbf{Q}_p, F^{00}V\big) \arrow[r] & H^1(\mathbf{Q}_p, W)\,. \arrow[uuuu, hook]
\end{tikzcd}$$

Lemma 1.7.

(a) $\dim_K H^{\mathrm{exc}}_{\mathrm{loc}} = 1$,

(b) $H^{\mathrm{exc}}_{\mathrm{loc}} \cap H^1_{\mathrm{nr}}(\mathbf{Q}_p, W) = 0$.

[1]This is the subspace denoted $\widetilde{\mathbf{T}}$ in [G94]. Page 161 of *loc. cit.* shows that it is one-dimensional.

Proof. This follows immediately from the definitions of $H^{\mathrm{exc}}_{\mathrm{glob}}$ and of $\overline{L}_p(V)$, together with assumption (U). $\square$

There are canonical coordinates on $H^1(\mathbf{Q}_p, W) \cong \mathrm{Hom}(G_{\mathbf{Q}_p}, W)$ given as follows. Every homomorphism $\varphi : G_{\mathbf{Q}_p} \to W$ factors through the maximal pro-p quotient of $G^{\mathrm{ab}}_{\mathbf{Q}_p}$, which is $\mathrm{Gal}(\mathbf{F}_\infty/\mathbf{Q}_p)$, where $\mathbf{F}_\infty$ is the compositum of two $\mathbf{Z}_p$-extensions of $\mathbf{Q}_p$: the cyclotomic one, $\mathbf{Q}_{p,\infty}$, and the maximal unramified abelian extension $\mathbf{Q}_p^{\mathrm{nr}}$. Let

$$\Gamma_\infty := \mathrm{Gal}(\mathbf{Q}_{p,\infty}, \mathbf{Q}_p) \cong \mathrm{Gal}(\mathbf{F}_\infty, \mathbf{Q}_p^{\mathrm{nr}})$$

and

$$\Gamma_{\mathrm{nr}} := \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{nr}}, \mathbf{Q}_p) \cong \mathrm{Gal}(\mathbf{F}_\infty, \mathbf{Q}_{p,\infty}),$$

then

$$\mathrm{Gal}(\mathbf{F}_\infty, \mathbf{Q}_p) = \Gamma_\infty \times \Gamma_{\mathrm{nr}}.$$

Therefore, $H^1(\mathbf{Q}_p, W)$ breaks up into $\mathrm{Hom}(\Gamma_\infty, W) \times \mathrm{Hom}(\Gamma_{\mathrm{nr}}, W)$. We have

$$\mathrm{Hom}(\Gamma_\infty, W) = \mathrm{Hom}(\Gamma_\infty, \mathbf{Q}_p) \otimes W$$

and

$$\mathrm{Hom}(\Gamma_{\mathrm{nr}}, W) = \mathrm{Hom}(\Gamma_{\mathrm{nr}}, \mathbf{Q}_p) \otimes W.$$

Composing the p-adic logarithm with the cyclotomic character provides a natural basis of $\mathrm{Hom}(\Gamma_\infty, \mathbf{Q}_p)$, and the function $\mathrm{ord}_p : \mathrm{Frob}_p \mapsto 1$ provides a natural basis of $\mathrm{Hom}(\Gamma_{\mathrm{nr}}, \mathbf{Q}_p)$. Coordinates are then provided by the isomorphisms

$$\begin{array}{rcl} \mathrm{Hom}(\Gamma_\infty, W) & \to & W \\ \log_p \chi_p \otimes w & \mapsto & w \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Hom}(\Gamma_{\mathrm{nr}}, W) & \to & W \\ \mathrm{ord}_p \otimes w & \mapsto & w. \end{array}$$

The L-invariant of V is the slope of $H^{\mathrm{exc}}_{\mathrm{loc}}$ with respect to these coordinates.

Specifically, we will compute the L-invariant in section 3 by constructing a global class $[c] \in H^1(\mathbf{Q}, V)$ satisfying

(CL1) $[c_v] \in H^1_{\mathrm{nr}}(\mathbf{Q}_v, V)$ for all $v \in \Sigma \setminus \{p\}$,

(CL2) $[c_p] \in F^{00} H^1(\mathbf{Q}_p, V)$,

(CL3) $[c_p] \not\equiv 0 \bmod F^1 V$.

This class then generates $H^{\mathrm{exc}}_{\mathrm{glob}}$, so its image $[\overline{c}_p] \in H^1(\mathbf{Q}_p, W)$ generates $H^{\mathrm{exc}}_{\mathrm{loc}}$. Let $u \in \mathbf{Z}_p^\times$ be any principal unit, so that under our normalizations, $\chi_p(\mathrm{rec}(u)) = u^{-1}$. Then, the coordinates of $[\overline{c}_p]$ are given by

$$\left(-\frac{1}{\log_p u} \overline{c}_p(\mathrm{rec}(u)), \overline{c}_p(\mathrm{Frob}_p) \right) \tag{1.3}$$

where $\overline{c}_p$ is a cocyle in $[\overline{c}_p]$. Note that these coordinates are independent of the choice of u. We then have the following formula for the L-invariant of V:

$$\mathcal{L}(V) = \frac{\overline{c}_p(\mathrm{Frob}_p)}{-\frac{1}{\log_p u}\overline{c}_p(\mathrm{rec}(u))}. \tag{1.4}$$

1.3 Symmetric power L-invariants of ordinary cusp forms

Let f be a p-ordinary,[2] holomorphic, cuspidal, normalized newform of weight $k \geq 2$, level $\Gamma_1(N)$ (prime to p), and trivial character. Let $E = \mathbf{Q}(f)$ be the field generated by the Fourier coefficients of f. Let $\mathfrak{p}_0|p$ be the prime of E above p corresponding to the fixed embedding ι_p, and let $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}(V_f)$ be the contragredient of the $\mathfrak{p}_0$-adic Galois representation (occurring in étale cohomology) attached to f by Deligne [D71] on the two-dimensional vector space V_f over $K := E_{\mathfrak{p}_0}$. Let α_p denote the root of $x^2 - a_p x + p^{k-1}$ which is a p-adic unit. The p-ordinarity assumption implies that

$$\rho_f|_{G_p} \sim \begin{pmatrix} \chi_p^{k-1}\delta^{-1} & \varphi \\ 0 & \delta \end{pmatrix}$$

where δ is the unramified character sending Frob_p to α_p ([W88, Theorem 2.1.4]). Thus, ρ_f is ordinary. Note that assumption (S) is automatically satisfied by all (Tate twists of) symmetric powers of ρ_f since all graded pieces of the ordinary filtration are one-dimensional. For condition (U) to be violated, we have must have $k = 2$ and $\alpha_p = 1$, but the Hasse bound shows that this is impossible.

Lemma 1.8. *If $(\mathrm{Sym}^n\rho_f)(r)$ is an exceptional, critical Tate twist of ρ_f, then $n \equiv 2 \pmod 4$, $r = \frac{n}{2}(1-k)$ or $\frac{n}{2}(1-k)+1$, and k is even. Furthermore, the exceptional subquotient is isomorphic to K or $K(1)$, respectively.*

Proof. The critical Tate twists are listed in [RgS08, Lemma 3.3]. Determining those that are exceptional is a quick computation, noting that δ is non-trivial. □

For the Tate twist by $\frac{n}{2}(1-k)+1$, the exceptional subquotient is isomorphic to $K(1)$, a case we did not treat in the previous section. However, Greenberg defines the L-invariant of such a representation as the L-invariant of its Tate dual, whose exceptional subquotient is isomorphic to the trivial representation. In fact, the Tate dual of the twist by $\frac{n}{2}(1-k)+1$ is the twist by $\frac{n}{2}(1-k)$. Accordingly, let m be a positive odd integer, $n := 2m$, $\rho_n := (\mathrm{Sym}^n\rho_f)(m(1-k))$, and assume k is even. We present a basic setup for computing Greenberg's L-invariant $\mathcal{L}(\rho_n)$ using a deformation of $\rho_m := \mathrm{Sym}^m\rho_f$. The main obstacle in carrying out this computation is to find a "sufficiently rich" deformation of ρ_m to obtain a non-trivial answer. We do so below in the case $n = 6$ for non-CM f (of weight ≥ 4) by transferring ρ_3 to $\mathrm{GSp}(4)_{/\mathbf{Q}}$ and using a Hida deformation on this group. The case $n = 2$ has been dealt with by Hida in [Hi04] (see also [H-PhD, Chapter 2]).

We need a lemma from the finite-dimensional representation theory of GL(2) whose proof we leave to the reader.

[2] More specifically, ι_∞-ordinary, in the sense that $\mathrm{ord}_p(\iota_\infty^{-1}(a_p)) = 0$, where a_p is the pth Fourier coefficient of f.

Lemma 1.9. *Let* Std *denote the standard representation of* GL(2)*. Then, for m an odd positive integer, there is a decomposition*

$$\mathrm{End}\left(\mathrm{Sym}^m\mathrm{Std}\right) \cong \bigoplus_{i=0}^{m} \left(\mathrm{Sym}^{2i}\mathrm{Std}\right) \otimes \det{}^{-i}.$$

Since $\det\rho_f = \chi_p^{k-1}$, this lemma implies that ρ_n occurs as a (global) direct summand in $\mathrm{End}\,\rho_m$. A deformation of ρ_m provides a class in $H^1(\mathbf{Q}, \mathrm{End}\,\rho_m)$. If its projection to $H^1(\mathbf{Q}, \rho_n)$ is non-trivial (and satisfies conditions (CL1–3) of the previous section), then it generates $H^{\mathrm{exc}}_{\mathrm{glob}}$ and can be used to compute $\mathcal{L}(\rho_n)$.

An obvious choice of deformation of ρ_m is the symmetric mth power of the Hida deformation of ρ_f. The cohomology class of this deformation has a non-trivial projection to $H^1(\mathbf{Q}, \rho_n)$ only when $m = 1$ (i.e. $n = 2$, the symmetric square). For larger m, a “richer” deformation is required. The aims of the remaining sections of this paper are to obtain such a deformation in the case $m = 3$ ($n = 6$) and to use it to find a formula for the L-invariant of ρ_6 in terms of derivatives of Frobenius eigenvalues varying in the deformation.

2 Input from $\mathrm{GSp}(4)_{/\mathbf{Q}}$

We use this section to set up our notations and conventions concerning the group $\mathrm{GSp}(4)_{/\mathbf{Q}}$, its automorphic representations, its Hida theory, and the Ramakrishnan–Shahidi symmetric cube lift from $\mathrm{GL}(2)_{/\mathbf{Q}}$ to it. We only provide what is required for our calculation of the L-invariant of ρ_6.

2.1 Notation and conventions

Let V be a four-dimensional vector space over $\mathbf{Q}$ with basis $\{e_1, \ldots, e_4\}$ equipped with the symplectic form given by

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Let $\mathrm{GSp}(4)$ be the group of symplectic similitudes of (V, J), i.e. $g \in \mathrm{GL}(4)$ such that ${}^t gJg = \nu(g)J$ for some $\nu(g) \in \mathbf{G}_m$. The stabilizer of the isotropic flag $0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle$ is the Borel subgroup B of $\mathrm{GSp}(4)$ whose elements are of the form

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & \frac{c}{b} & * \\ & & & \frac{c}{a} \end{pmatrix}.$$

Writing an element of the maximal torus T as

$$t = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \frac{\nu(t)}{t_2} & \\ & & & \frac{\nu(t)}{t_1} \end{pmatrix},$$

we identify the character group $X^*(T)$ with triples (a, b, c) satisfying $a + b \equiv c \pmod 2$ so that

$$t^{(a,b,c)} = t_1^a t_2^b \nu(t)^{(c-a-b)/2}.$$

The dominant weights with respect to B are those with $a \geq b \geq 0$. If Π is an automorphic representation of $\mathrm{GSp}(4, \mathbf{A})$ whose infinite component Π_∞ is a holomorphic discrete series, we will say Π has weight (a, b) if Π_∞ has the same infinitesimal character as the algebraic representation of $\mathrm{GSp}(4)$ whose highest weight is (a, b, c) (for some c). For example, a classical Siegel modular form of (classical) weight (k_1, k_2) gives rise to an automorphic representation of weight $(k_1 - 3, k_2 - 3)$ under our normalizations.

2.2 The Ramarkrishnan–Shahidi symmetric cube lift

We wish to move the symmetric cube of a cusp form f to a cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbf{A})$ in order use the Hida theory on this group to obtain an interesting Galois deformation of the symmetric cube of ρ_f. The following functorial transfer due to Ramakrishnan and Shahidi ([RS07, Theorem A′]) allows us to do so in certain circumstances.

Theorem 2.1 (Ramakrishnan–Shahidi [RS07])**.** *Let π be the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbf{A})$ defined by a holomorphic, non-CM newform f of even weight $k \geq 2$, level N, and trivial character. Then, there is an irreducible cuspidal automorphic representation Π of $\mathrm{GSp}(4, \mathbf{A})$ with the following properties*

(a) Π_∞ *is in the holomorphic discrete series with its L-parameter being the symmetric cube of that of π,*

(b) Π *has weight $(2(k-2), k-2)$, trivial central character, and is unramified outside of N,*

(c) $\Pi^K \neq 0$ *for some compact open subgroup K of $\mathrm{GSp}(4, \mathbf{A}_f)$ of level equal to the conductor of* $\mathrm{Sym}^3 \rho_f$,

(d) $L(s, \Pi) = L(s, \pi, \mathrm{Sym}^3)$, *where $L(s, \Pi)$ is the dergree 4 spin L-function,*

(e) Π *is weakly equivalent[3] to a globally generic cuspidal automorphic representation,*

[3] Recall that "weakly equivalent" means that the local components are isomorphic for almost all places.

(f) Π *is not CAP, nor endoscopic.*[4]

We remark that the weight in part (b) can be read off from the L-parameter of Π_∞ given in [RS07, (1.7)]. As for part (e), note that the construction of Π begins by constructing a globally generic representation on the bottom of page 323 of [RS07], and ends by switching, in the middle of page 326, the infinite component from the generic discrete series element of the archimedean L-packet to the holomorphic one. Alternatively, in [Wei08], Weissauer has shown that any non-CAP non-endoscopic irreducible cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbf{A})$ is weakly equivalent to a globally generic cuspidal automorphic representation.

2.3 Hida deformation of ρ_3 on $\mathrm{GSp}(4)_{/\mathbf{Q}}$

Let f a p-ordinary, holomorphic, non-CM, cuspidal, normalized newform of even weight $k \geq 4$, level $\Gamma_1(N)$ (prime to p), and trivial character. We have added the non-CM hypothesis to be able to use the Ramakrishnan–Shahidi lift.[5] According to lemma 1.8, we only need to consider even weights. The restriction $k \neq 2$ is forced by problems with the Hida theory on $\mathrm{GSp}(4)_{/\mathbf{Q}}$ in the weight $(0,0)$.

Tilouine and Urban ([TU99], [U01], [U05]), as well as Pilloni ([Pi10], building on Hida ([Hi02])) have worked on developing Hida theory on $\mathrm{GSp}(4)_{/\mathbf{Q}}$. In this section, we describe the consequences their work has on the deformation theory of $\rho_3 = \mathrm{Sym}^3\rho_f$ (where ρ_f is as described in section 1.3).

We begin by imposing two new assumptions:

(Ét) the universal ordinary p-adic Hecke algebra of $\mathrm{GSp}(4)_{/\mathbf{Q}}$ is étale over the Iwasawa algebra at the height one prime corresponding to Π;

(RAI) the representation ρ_3 is residually absolutely irreducible.

Considering ρ_3 as the p-adic Galois representation attached to the Ramakrishnan–Shahidi lift Π of f, we obtain a ring $\mathcal{A}$ of p-adic analytic functions in two variables (s_1,s_2) on some neighbourhood of the point $(a,b) = (2(k-2), k-2) \in \mathbf{Z}_p^2$, a free rank four module $\mathcal{M}$ over $\mathcal{A}$, and a deformation $\widetilde{\rho}_3 : G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathcal{A}}(\mathcal{M})$ of ρ_3 such that $\widetilde{\rho}_3(a,b) = \rho_3$ and

$$\widetilde{\rho}_3|_{G_p} \sim \begin{pmatrix} \theta_1\theta_2\mu_1 & \xi_{12} & \xi_{13} & \xi_{14} \\ & \theta_2\mu_2 & \xi_{23} & \xi_{24} \\ & & \theta_1\mu_2^{-1} & \xi_{34} \\ & & & \mu_1^{-1} \end{pmatrix} \tag{2.1}$$

[4] Recall that an irreducible, cuspidal, automorphic representation of $\mathrm{GSp}(4,\mathbf{A})$ is "CAP" if it is weakly equivalent to the induction of an automorphic representation on a proper Levi subgroup, and it is "endoscopic" if the local L-factors of its spin L-function are equal, at almost all places, to the product of the local L-factors of two cuspidal automorphic representations of $\mathrm{GL}(2,\mathbf{A})$ with equal central characters.

[5] This is not really an issue as the CM case is much simpler.

where the μ_i are unramified, and

$$\mu_1(a,b) = \delta^{-3} \tag{2.2}$$
$$\mu_2(a,b) = \delta^{-1} \tag{2.3}$$
$$\theta_1(s_1,s_2) = \chi_p^{s_2+1} \tag{2.4}$$
$$\theta_1(a,b) = \chi_p^{k-1} \tag{2.5}$$
$$\theta_2(s_1,s_2) = \chi_p^{s_1+2} \tag{2.6}$$
$$\theta_2(a,b) = \chi_p^{2(k-1)}. \tag{2.7}$$

Remark 2.2. Assumption (RAI) allows us to take the integral version of [TU99, Theorem 7.1] (see the comment of *loc. cit.* at the end of §7) and assumption (Ét) says that the coefficients are p-adic analytic. The shape of $\widetilde{\rho}_3|_{G_p}$ can be seen as follows. That four distinct Hodge–Tate weights show up can be seen by using [U01, Lemma 3.1] and the fact that both Π and the representation obtained from Π by switching the infinite component are automorphic. Applying Theorem 3.4 of *loc. cit.* gives part of the general form of $\widetilde{\rho}_3|_{G_p}$ (taking into account that we work with the contragredient). The form the unramified characters on the diagonal take is due to $\widetilde{\rho}_3|_{G_p}$ taking values in the Borel subgroup B (this follows from Corollary 3.2 and Proposition 3.4 of *loc. cit.*). That the specializations of the μ_i and θ_i at (a,b) are what they are is simply because $\widetilde{\rho}_3$ is a deformation of ρ_3.

We may take advantage of assumption (Ét) to determine a bit more information about the μ_i. Indeed, let $\widetilde{\rho}_f$ denote the Hida deformation (on $\mathrm{GL}(2)_{/\mathbf{Q}}$) of ρ_f, so that

$$\widetilde{\rho}_f|_{G_p} \sim \begin{pmatrix} \theta\mu^{-1} & \xi \\ 0 & \mu \end{pmatrix}$$

where θ, μ, and ξ are p-adic analytic functions on some neighbourhood of $s = k$, $\theta(s) = \chi_p^{s-1}$, and $\mu(s)$ is the unramified character sending Frob_p to $\alpha_p(s)$ (where $\alpha_p(s)$ is the p-adic analytic function giving the pth Fourier coefficients in the Hida family of f) ([W88, Theorem 2.2.2]). By [GhVa04, Remark 9], we know that every arithmetic specialization of $\widetilde{\rho}_f$ is non-CM. We may thus apply the Ramakrishnan–Shahidi lift to the even weight specializations and conclude that $\mathrm{Sym}^3\widetilde{\rho}_f$ is an ordinary modular deformation of ρ_3. Assumption (Ét) then implies that $\mathrm{Sym}^3\widetilde{\rho}_f$ is a specialization of $\widetilde{\rho}_3$. Since the weights of the symmetric cube lift of a weight k' cusp form are $(2(k'-2), k'-2)$, we can conclude that $\mathrm{Sym}^3\widetilde{\rho}_f$ is the "sub-family" of $\widetilde{\rho}_3$ where $s_1 = 2s_2$. Thus,

$$\mu_1(2s,s) = \mu^{-3}(s+2)$$
$$\mu_2(2s,s) = \mu^{-1}(s+2).$$

Applying the chain rule yields

$$2\partial_1\mu_1(a,b) + \partial_2\mu_1(a,b) = -\frac{3\mu'(k)}{\delta^4} \tag{2.8}$$

$$2\partial_1\mu_2(a,b) + \partial_2\mu_2(a,b) = -\frac{\mu'(k)}{\delta^2}. \tag{2.9}$$

3 Calculating the L-invariant

For the remainder of this article, let f a p-ordinary, holomorphic, non-CM, cuspidal, normalized newform of even weight $k \geq 4$, level $\Gamma_1(N)$ (prime to p), and trivial character. Let ρ_f, ρ_3, and ρ_6 be as in section 1.3, and let W denote the exceptional subquotient of ρ_6. Furthermore, assume condition (Z) that $\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_6) = 0$. We now put together the ingredients of the previous sections to compute Greenberg's L-invariant of ρ_6.

3.1 Constructing the global Galois cohomology class

Recall that if ρ_3' is an infinitesimal deformation of ρ_3 (over $K[\epsilon] := K[x]/(x^2)$), a corresponding cocycle $c_3' : G_{\mathbf{Q}} \to \operatorname{End}\rho_3$ is defined by the equation

$$\rho_3'(g) = \rho_3(g)(1 + \epsilon c_3'(g)).$$

Let $\widetilde{\rho}_3$ be the deformation of ρ_3 constructed in section 2.3. Taking a first order expansion of the entries of $\widetilde{\rho}_3$ around $(a,b) = (2(k-2), k-2))$ in any given direction yields an infinitesimal deformation of ρ_3. We parametrize these as follows. A p-adic analytic function $F \in \mathcal{A}$ has a first-order expansion near $(s_1, s_2) = (a,b)$ given by

$$F(a+\epsilon_1, b+\epsilon_2) \approx F(a,b) + \epsilon_1\partial_1 F(a,b) + \epsilon_2\partial_2 F(a,b).$$

We introduce a parameter Δ such that $\epsilon_1 = (1-\Delta)\epsilon$ and $\epsilon_2 = \Delta\epsilon$. Let $\widetilde{\rho}_{3,\Delta}$ denote the infinitesimal deformation of ρ_3 obtained by first specializing $\widetilde{\rho}_3$ along the direction given by Δ, then taking the quotient by the ideal $((s_1,s_2)-(a,b))^2$. Let $c_{6,\Delta}$ denote the projection of its corresponding cocyle to ρ_6 (in the decomposition of lemma 1.9).

3.2 Properties of the global Galois cohomology class

To use $c_{6,\Delta}$ to compute the L-invariant of ρ_6, we must show that it satisfies conditions (CL1–3) of section 1.2. The proofs of lemma 1.2 and 1.3 of [Hi07] apply to the cocycle $c_{6,\Delta}$ to show that it satisfies (CL1).[6]

To verify conditions (CL2) and (CL3) (and to compute the L-invariant of ρ_6), we need to find

[6] The deformation $\widetilde{\rho}_3$ clearly satisfies conditions (K$_3$1–4) of [Hi07, §0] and our $c_{6,\Delta}$ is a special case of the cocycles Hida defines in the proof of lemma 1.2 of *loc. cit.*

an explicit formula for part of $c_{6,\Delta}|_{G_p}$. We know that

$$\rho_f|_{G_p} \sim \begin{pmatrix} \chi_p^{k-1}\delta^{-1} & \varphi \\ 0 & \delta \end{pmatrix}.$$

Taking the symmetric cube (considered as a subspace of the third tensor power) yields

$$\rho_3|_{G_p} \sim \begin{pmatrix} \chi_p^{3(k-1)}\delta^{-3} & \frac{3\chi_p^{2(k-1)}\varphi}{\delta^2} & \frac{3\chi_p^{k-1}\varphi^2}{\delta} & \varphi^3 \\ & \chi_p^{2(k-1)}\delta^{-1} & 2\chi_p^{k-1}\varphi & \delta\varphi^2 \\ & & \chi_p^{k-1}\delta & \delta^2\varphi \\ & & & \delta^3 \end{pmatrix}.$$

Taking first-order expansions of the entries of $\widetilde{\rho}_3\rho_3^{-1} - I_4$, specializing along the direction given by Δ, and projecting yields $c_{6,\Delta}$. However, since we are interested in an explicit formula for $c_{6,\Delta}|_{G_p}$, we need to determine a basis that gives the decomposition of lemma 1.9. This can be done using the theory of raising and lowering operators. We obtain the following result.

Theorem 3.1. *In such an aforementioned basis,*

$$c_{6,\Delta}|_{G_p} \sim \begin{pmatrix} * \\ (1-\Delta)\left(\frac{\partial_1\theta_2}{\chi_p^{2(k-1)}} - \frac{2\partial_1\theta_1}{\chi_p^{k-1}} - \delta^3\partial_1\mu_1 + 3\delta\partial_1\mu_2\right) \\ +\Delta\left(\frac{\partial_2\theta_2}{\chi_p^{2(k-1)}} - \frac{2\partial_2\theta_1}{\chi_p^{k-1}} - \delta^3\partial_2\mu_1 + 3\delta\partial_2\mu_2\right) \\ 0 \end{pmatrix} \tag{3.1}$$

where $*$ and 0 are both 3×1, and all derivatives are evaluated at (a, b).

Since the bottom three coordinates in (3.1) are zero, the image of $c_{6,\Delta}|_{G_p}$ lands in $F^{00}\rho_6$, i.e. $c_{6,\Delta}$ satisfies (CL2). If we can show that the middle coordinate is non-zero, then $c_{6,\Delta}$ satisfies (CL3). In fact, we will show that $c_{6,\Delta}$ satisfies (CL3) if, and only if, $\Delta \neq 1/3$ (in this latter case, we will show that $[c_{6,1/3}] = 0$).

Let $\overline{c}_{6,\Delta}$ denote the image of $c_{6,\Delta}$ in $H^1(\mathbf{Q}_p, W)$. Let

$$\alpha_p^{(i,j)} := \partial_j\mu_i(a,b)(\mathrm{Frob}_p).$$

Corollary 3.2. *The coordinates of* $\overline{c}_{6,\Delta}$*, as in equation* (1.3)*, are*

$$\left(1-3\Delta,(1-\Delta)\left(-\alpha_p^3\alpha_p^{(1,1)}+3\alpha_p\alpha_p^{(2,1)}\right)+\Delta\left(-\alpha_p^3\alpha_p^{(1,2)}+3\alpha_p\alpha_p^{(2,2)}\right)\right).$$

In particular, if $\Delta\neq 1/3$*, then* $c_{6,\Delta}$ *satisfies (CL3).*

Before proving this, we state and prove a lemma.

Lemma 3.3. *Recall that* $\theta(s)=\chi_p^{s-1}$*. For any integer* $s\geq 2$*, and any principal unit* u*,*

(a) $\theta'(s)(\mathrm{Frob}_p)=0$*,*

(b) $\dfrac{\theta'(s)(\mathrm{rec}(u))}{\chi_p^{s-1}(\mathrm{rec}(u))}=-\log_p u$*.*

Proof. The first equality is simply because $\chi_p(\mathrm{Frob}_p)=1$. For the second, recall that $\chi_p(\mathrm{rec}(u))=u^{-1}$, so $\theta(s)(\mathrm{rec}(u))=u^{1-s}$. Thus, the logarithmic derivative of $\theta(s)(\mathrm{rec}(u))$ is indeed $-\log_p u$. □

Proof of corollary 3.2. The first coordinate is obtained by taking an arbitrary principal unit u, evaluating $\overline{c}_{6,\Delta}$ and $\mathrm{rec}(u)$ and dividing by $-\log_p u$. By equations (2.4) and (2.6), $\partial_i\theta_i=0$. Combining the fact that the μ_i are unramified with part (b) of the above lemma yields

$$\frac{\overline{c}_{6,\Delta}(\mathrm{rec}(u))}{-\log_p u}=\frac{(1-\Delta)(-\log_p u)+\Delta(-2\log_p u)}{-\log_p u}=1-3\Delta.$$

If $\Delta\neq 1/3$, the first coordinate is non-zero, so $\overline{c}_{6,\Delta}$ itself is non-zero, so $c_{6,\Delta}$ satisfies (CL3).

Combining part (a) of the above lemma with equations (2.2) and (2.3) yields the second coordinate (recall that $\delta(\mathrm{Frob}_p)=\alpha_p$). □

Remark 3.4. If we take $\Delta=1/3$, the first coordinate of $\overline{c}_{6,1/3}$ vanishes. Hence, $\overline{c}_{6,1/3}\in H^1_{\mathrm{nr}}(\mathbf{Q}_p,W)$. Therefore, $[c_{6,1/3}]\in\overline{\mathrm{Sel}}_{\mathbf{Q}}(V)=0$ (by assumption (Z)). The direction $\Delta=1/3$ is the one for which $\epsilon_1/\epsilon_2=2$, i.e. the direction corresponding to the symmetric cube of the GL(2) Hida deformation of ρ_f. This is an instance of the behaviour mentioned at the end of section 1.3.

3.3 Formula for the L-invariant

Tying all this together yields the main theorem of this article.

Theorem A. *Let* $p\geq 3$ *be a prime. Let* f *a* p*-ordinary, holomorphic, non-CM, cuspidal, normalized newform of even weight* $k\geq 4$*, level* $\Gamma_1(N)$ *(prime to* p*), and trivial character. Let* ρ *be a critical, exceptional Tate twist of* $\mathrm{Sym}^6\rho_f$*, i.e.* $\rho=\rho_6=(\mathrm{Sym}^6\rho_f)(3(1-k))$ *or its Tate dual. Assume conditions* (Ét)*,* (RAI)*, and* (Z)*. Then,*

$$\mathcal{L}(\rho)=-\alpha_p^3\alpha_p^{(1,1)}+3\alpha_p\alpha_p^{(2,1)}. \tag{3.2}$$

Proof. Pick any $\Delta \neq 1/3$. We've shown that $[c_{6,\Delta}]$ satisfies (CL1–3) and hence generates $H^{\mathrm{exc}}_{\mathrm{glob}}$. The coordinates of its image in $H^1(\mathbf{Q}_p, W)$ were obtained in corollary 3.2. Therefore, $\mathcal{L}(\rho_6)$ can be computed from equation (1.4). Specifically, the result is obtained by solving the system of linear equations in $\mathcal{L}(\rho_6)$ and the $\alpha_p^{(i,j)}$ given by the coordinates of $\overline{c}_{6,\Delta}$ and equations (2.8) and (2.9). The L-invariant of ρ_6^* is by definition that of ρ_6. $\square$

Remark 3.5. We could express this result in terms of other $\alpha_p^{(i,j)}$. For example, picking $\Delta = 1$ yields

$$\mathcal{L}(\rho_6) = \frac{1}{2}\alpha_p^3\alpha_p^{(1,2)} - \frac{3}{2}\alpha_p\alpha_p^{(2,2)}.$$

3.4 Relation to Greenberg's L-invariant of the symmetric square

We can carry out the above analysis for the projection to $\rho_2 := (\mathrm{Sym}^2\rho_f)(1-k)$ in lemma 1.9 and compare the value of $\mathcal{L}(\rho_2)$ obtained with the known value ([Hi04, Theorem 1.1], [H-PhD, Theorem A])

$$\mathcal{L}(\rho_2) = -2\frac{\alpha_p'}{\alpha_p}$$

where $\alpha_p' = \mu'(k)(\mathrm{Frob}_p) = \alpha_p'(k)$, and one assumes that $\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_2) = 0$.[7] The restriction of the cocycle $c_{2,\Delta}$ (in an appropriate basis) is

$$c_{2,\Delta}|_{G_p} \sim \begin{pmatrix} * \\ (1-\Delta)\left(-\frac{2\partial_1\theta_2}{\chi_p^{2(k-1)}} - \frac{\partial_1\theta_1}{\chi_p^{k-1}} - 3\delta^3\partial_1\mu_1 - \delta\partial_1\mu_2\right) \\ +\Delta\left(-\frac{2\partial_2\theta_2}{\chi_p^{2(k-1)}} - \frac{\partial_2\theta_1}{\chi_p^{k-1}} - 3\delta^3\partial_2\mu_1 - \delta\partial_2\mu_2\right) \\ 0 \end{pmatrix}$$

Accordingly, the coordinates of the class $\overline{c}_{2,\Delta}$ are

$$\left(\Delta - 2, (1-\Delta)\left(-3\alpha_p^3\alpha_p^{(1,1)} - \alpha_p\alpha_p^{(2,1)}\right) + \Delta\left(-3\alpha_p^3\alpha_p^{(1,2)} - \alpha_p\alpha_p^{(2,2)}\right)\right).$$

The cocycle $c_{2,\Delta}$ can be used to compute $\mathcal{L}(\rho_2)$ when $\Delta \neq 2$. When $\Delta = 2$, one has, as above, $[c_{2,\Delta}] \in \overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_2)$. Taking $\Delta = 0$ yields

$$\mathcal{L}(\rho_2) = \frac{3}{2}\alpha_p^3\alpha_p^{(1,1)} + \frac{1}{2}\alpha_p\alpha_p^{(2,1)}. \tag{3.3}$$

Combining equations (3.2) and (3.3) yields the following relation between L-invariants.

[7] This vanishing is known in many cases due to work of Hida ([Hi04]), Kisin ([Ki04]), and Weston ([Wes04]). See those papers for details or [H-PhD, Theorem 2.1.1] for a summary.

Theorem B. *Assuming* (Ét), (RAI), (Z), *and* $\overline{\mathrm{Sel}}_{\mathbf{Q}}(\rho_2) = 0$, *we have*

$$\mathcal{L}(\rho_6) = -10\alpha_p^3\alpha_p^{(1,1)} + 6\mathcal{L}(\rho_2).$$

Remark 3.6. There is a guess, suggested by Greenberg [G94, p. 170], that the L-invariants of all symmetric powers of ρ_f should be equal. This is known in the cases where it is relatively easy to compute the L-invariant, namely when f corresponds to an elliptic curve with split, multiplicative reduction at p, or when f has CM. In the case at hand, we fall one relation short of showing the equality of $\mathcal{L}(\rho_6)$ and $\mathcal{L}(\rho_2)$. Equality would occur if one knew the relation

$$\alpha_p^{(1,1)} \stackrel{?}{=} -\frac{\alpha_p'}{\alpha_p^4}.$$

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